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On equivalence of moduli of smoothness of polynomials in L_p, $0 < p \leqslant \infty$ [☆]

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Abstract

It is well known that $\omega^r(f,t)_p \leqslant t\,\omega^{r-1}(f',t)_p \leqslant t^2\omega^{r-2}(f'',t)_p \leqslant \cdots$ for functions $f \in \mathbf{W}_p^r$, $1 \leqslant p \leqslant \infty$. For general functions $f \in \mathbf{L}_p$, it does not hold for $0 < p < 1$, and its inverse is not true for any p in general. It has been shown in the literature, however, that for certain classes of functions the inverse is true, and the terms in the inequalities are all equivalent. Recently, Zhou and Zhou proved the equivalence for polynomials with $p = \infty$. Using a technique by Ditzian, Hristov and Ivanov, we give a simpler proof to their result and extend it to the $\mathbf{L}_p$ space for $0 < p \leqslant \infty$. We then show its analogues for the Ditzian–Totik modulus of smoothness $\omega_\varphi^r(f,t)_p$ and the weighted Ditzian–Totik modulus of smoothness $\omega_\varphi^r(f,t)_{w,p}$ for polynomials with $\varphi(x) = \sqrt{1-x^2}$.

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1. Introduction

Throughout this paper we denote by $\|\cdot\|_{\mathbf{L}_p[a,\,b]}$ the usual $\mathbf{L}_p$ norm (quasi-norm if $p < 1$) on the interval $[a,\ b]$ for $0 < p < \infty$, and the uniform norm for $p = \infty$. If there is no possibility of confusion, we will use $\|\cdot\|_p$ for $\|\cdot\|_{\mathbf{L}_p[-1,\,1]}$, and $\|\cdot\|$ for $\|\cdot\|_{\mathbf{L}_\infty[-1,\,1]}$. We define the symmetric difference operator Δ_h by

$$\Delta_h(f, x) := f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

and Δ_h^r by

$$\Delta_h^r f(x) := \Delta_h(\Delta_h^{r-1}(f, x)) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \frac{rh}{2} - kh\right). \tag{1.1}$$

Similarly we define the forward difference operator by

$$\overrightarrow{\Delta}_h^r(f, x) := \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x + kh) \tag{1.2}$$

and the backward difference operator by

$$\overleftarrow{\Delta}_h^r(f, x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x - kh). \tag{1.3}$$

For any $f \in \mathbf{L}_p[a,\ b]$ and $t \geqslant 0$, let

$$\begin{aligned}\omega^r(f, t)_p &:= \sup_{0 \leqslant h \leqslant t} \|\Delta_h^r f\|_{\mathbf{L}_p[a+rh/2,\, b-rh/2]} = \sup_{0 \leqslant h \leqslant t} \|\overrightarrow{\Delta}_h^r f\|_{\mathbf{L}_p[a,\, b-rh]} \\ &= \sup_{0 \leqslant h \leqslant t} \|\overleftarrow{\Delta}_h^r f\|_{\mathbf{L}_p[a+rh,\, b]}\end{aligned}$$

be the usual rth modulus of smoothness of f, with $\omega^0(f, t)_p$ understood as $\|f\|_{\mathbf{L}_p[a,\,b]}$. We will omit the subscript ∞ in all moduli with $p = \infty$, for example $\omega^r(\cdot, \cdot) := \omega^r(\cdot, \cdot)_\infty$.

For $1 \leqslant p \leqslant \infty$, if $f \in \mathbf{W}_p^k[a,\ b]$, the Sobolev Space of functions f on $[a,\ b]$ such that $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in \mathbf{L}_p[a,\ b]$, it is well known that

$$\omega^r(f, t)_p \leqslant t\, \omega^{r-1}(f', t)_p \leqslant \cdots \leqslant \begin{cases} t^k \omega^{r-k}(f^{(k)}, t)_p, & r > k, \\ t^r \|f^{(r)}\|_{\mathbf{L}_p[a,\,b]}, & r \leqslant k. \end{cases} \tag{1.4}$$

The inverse of (1.4) with any constants independent of f and t is not true in general. One counterexample for $1 < p < \infty$ is given by $f(x) = (x + \varepsilon)^{1-1/p}$ on $[0,\ 1]$ with $0 < \varepsilon \leqslant 1$. It is readily to verify that $\omega^r(f, t)_p \leqslant C(r)$ for any $0 < t \leqslant 1$, but $\|f^{(l)}\|_{\mathbf{L}_p[0,\,1]} \to \infty$ as $\varepsilon \to 0^+$ for any $l \geqslant 1$. Yu and Zhou [18] proved in 1994 part of the inverse in a special case for splines, namely

$$u\, \omega^{m-1}(s', u) \leqslant C(m) \omega^m(s, u), \tag{1.5}$$

where s is any spline of order $m > 1$ with equally spaced knots, and u is the mesh size. Hu and Yu [9] proved in 1995 that for such splines s the whole inverse of (1.4) holds true for any $t \geqslant 0$ not exceeding u, thus

$$\omega^r(s,t)_p \sim t\,\omega^{r-1}(s',t)_p \sim t^2\omega^{r-2}(s'',t)_p \ldots, \quad 0 \leqslant t \leqslant u, \ \ 1 \leqslant p \leqslant \infty, \tag{1.6}$$

with the equivalence constants depending only on $\max(r,m)$. A few years later, Hu [6] generalized (1.6) to splines with any (fixed) knot sequence, and further to principal shift-invariant spaces and wavelets under certain conditions. Equivalence (1.6) for splines has played key roles in shape-preserving spline and polynomial approximation repeatedly, (see [7,8,10,11]), which motivates us to investigate further along the line. In fact, we believe similar results are valid for many classes of functions, univariate and multivariate.

It seems to us the whole topic of equivalence of moduli of smoothness, in the sense of (1.6), has been overlooked to a great extent. The first primitive result (1.5) appeared in 1994, many years after the theory of splines with fixed knots was established. The topic had not been explicitly discussed until 1995 [9], to our best knowledge. Some authors were close, sometimes extremely close, to results similar to (1.6), but failed to take the last step, or simply failed to claim them. One good example is the following theorem:

Theorem 1. *Let $n \geqslant 1$, $r \geqslant 1$ and $0 < p \leqslant \infty$. Then for $T_n \in \mathcal{T}_n$, the space of trigonometric polynomials on $[-\pi, \pi]$ of degree $\leqslant n$, we have*

$$\omega^r(T_n,t)_p \sim t\,\omega^{r-1}(T_n',t)_p \sim \cdots \sim t^r \|T_n^{(r)}\|_{\mathbf{L}_p[-\pi,\pi]}, \quad 0 < t \leqslant n^{-1}, \tag{1.7}$$

where the equivalence constants depend only on r and $q := \min(1, p)$.

For $p = \infty$, the theorem follows, as pointed out by Zhou and Zhou [19], from (1.4) and

$$\|T_n^{(r)}\|_{\mathbf{L}_\infty[-\pi,\pi]} \leqslant \left(\frac{n}{2\sin nh}\right)^r \|\Delta_{2h}^r T_n\|_{\mathbf{L}_\infty[-\pi,\pi]}, \qquad 0 < h < \frac{\pi}{n},$$

which has been known for long time (see [16]). As for $0 < p < \infty$, Ditzian et al., showed in the proof of Theorem 3.1 in [3] that for $0 < h \leqslant n^{-1}$

$$\|\Delta_h^r T_n\|_{\mathbf{L}_p[-\pi,\pi]} \leqslant C h^r \|T_n^{(r)}\|_{\mathbf{L}_p[-\pi,\pi]}, \tag{1.8a}$$

$$h^r \|T_n^{(r)}\|_{\mathbf{L}_p[-\pi,\pi]} \leqslant 2^{1/q} \|\Delta_h^r T_n\|_{\mathbf{L}_p[-\pi,\pi]}, \tag{1.8b}$$

where, and throughout the paper,

$$q := \min(p, 1). \tag{1.9}$$

The two inequalities immediately give $\omega^r(T_n,t)_p \sim t^r \|T_n^{(r)}\|_{\mathbf{L}_p[-\pi,\pi]}$, $0 < t \leqslant n^{-1}$, from which the other cases of the theorem follow, (by replacing r by $r-j$ and replacing T_n by $T_n^{(j)}$). [2]

[2] As one referee of this paper points out, the theorem also follows from Theorem 3.1 of [3] itself, rather than from its proof, by a standard argument.

Recently, Zhou and Zhou [19] proved the following analogue of (1.7) for $\mathcal{P}_n$, the space of algebraic polynomials of degree $\leqslant n$ (in a slightly different form).

Theorem A. *Let* $P_n \in \mathcal{P}_n[-1,\ 1]$, $n > r \geqslant 1$. *Then for any* $t \in [0,\ n^{-2}]$

$$\omega^r(P_n, t) \sim t\,\omega^{r-1}(P_n', t) \sim \cdots \sim t^r \|P_n^{(r)}\|, \qquad 0 \leqslant t \leqslant n^{-2}, \tag{1.10}$$

with the equivalence constants depending only on r.

In §3 we will generalize (1.10) to $\mathbf{L}_p$, $0 < p \leqslant \infty$, and then prove similar results for the Ditzian–Totik (DT) modulus ω_φ^r, the DT main-part modulus, and for the weighted DT modulus with a rather general weight function w. A technique similar to that in [3] will be used. The last section will be devoted to applications. But before all this, we need to introduce in the following section some notation, preliminaries, and a few inequalities of fundamental importance in algebraic polynomial approximation.

2. Notation and preliminaries

Throughout the paper the step-weight function is chosen as

$$\varphi(x) := \sqrt{1 - x^2} \tag{2.1}$$

unless otherwise mentioned. The DT modulus of smoothness [3] is defined by

$$\omega_\varphi^r(f, t)_p := \sup_{0 \leqslant h \leqslant t} \|\Delta_{h\varphi}^r f\|_{\mathbf{L}_p(I_{rh})},$$

where

$$I_{rh} := \left\{ x \in [-1,\ 1] : -1 \leqslant x - \frac{rh\varphi(x)}{2} \leqslant x + \frac{rh\varphi(x)}{2} \leqslant 1 \right\}.$$

If we write $I_{rh} = [-1 + h^{*2},\ 1 - h^{*2}]$, then simple computation shows

$$h^{*2} = \frac{2(\frac{rh}{2})^2}{1 + (\frac{rh}{2})^2} \quad \text{and} \quad \frac{rh}{2} \leqslant h^* \leqslant \frac{rh}{\sqrt{2}}.$$

Sometimes $\omega_\varphi^r(f, t)_p$ can be too sensitive to the values of the function near the endpoints, and its exact domain I_{rh} is difficult to calculate, thus the so-called main-part modulus of smoothness

$$\Omega_\varphi^r(f, t)_p := \sup_{0 \leqslant h \leqslant t} \|\Delta_{h\varphi}^r f\|_{\mathbf{L}_p[-1+2r^2h^2,\ 1-2r^2h^2]} \tag{2.2}$$

[3] The DT modulus of smoothness is defined in [5] for a class of step-weight functions φ, not only for $\sqrt{1 - x^2}$. Our results only involve $\varphi = \sqrt{1 - x^2}$.

has been introduced. It is defined on a smaller domain but preserves most of the "essential behavior" [5, §3.3]. If $P_n \in \mathcal{P}_n$, then by Taylor's Theorem

$$\begin{aligned}\Delta_h^r P_n(x) &= \sum_{k=0}^{r}(-1)^k \binom{r}{k} \sum_{j=0}^{n} \frac{P_n^{(j)}(x)}{j!}[(r/2-k)h]^j \\ &= \sum_{j=0}^{n} \frac{P_n^{(j)}(x)}{j!} h^j \sum_{k=0}^{r}(-1)^k \binom{r}{k} (r/2-k)^j.\end{aligned}$$

Denoting $g_j(x) := x^j$, we have

$$\sum_{k=0}^{r}(-1)^k \binom{r}{k} (r/2-k)^j = \Delta_1^r\, g_j(0) = \begin{cases} 0 & \text{if } j-r \text{ is odd or } j<r, \\ \dfrac{j!}{(j-r)!}\xi_j^{j-r} & \text{otherwise,}\end{cases}$$

where $-r/2 < \xi_j < r/2$ depends only on r and j. Combining all this we have

$$\Delta_h^r P_n(x) = \sum_{\substack{r\leqslant j\leqslant n \\ j-r \text{ even}}} \frac{P_n^{(j)}(x)}{(j-r)!} h^j \xi_j^{j-r} = \sum_{k=0}^{K} \frac{P_n^{(r+2k)}(x)}{(2k)!} h^{r+2k} \xi_{r+2k}^{2k}, \tag{2.3}$$

where $K := \lfloor \frac{n-r}{2} \rfloor$. Replacing h by $h\varphi(x)$ yields

$$\Delta_{h\varphi(x)}^r P_n(x) = \sum_{k=0}^{K} \frac{\varphi(x)^{r+2k} P_n^{(r+2k)}(x)}{(2k)!} h^{r+2k} \xi_{r+2k}^{2k}. \tag{2.4}$$

Note $\varphi^{r+2k} \in \mathcal{P}_{r+2k}$ if r is even, thus

$$\Delta_{h\varphi}^r P_n = \begin{cases} Q_n & \text{if } r \text{ is even} \\ \sqrt{1-x^2}\ Q_{n-1} & \text{if } r \text{ is odd,}\end{cases} \tag{2.5}$$

where $Q_m \in \mathcal{P}_m$, $m = n-1, n$. Similar calculation shows

$$\overrightarrow{\Delta}_h^r P_n(x) = \sum_{k=0}^{n-r} \frac{P_n^{(r+k)}(x)}{k!} h^{r+k} \xi_{r+k}^{k}, \qquad 0 < \xi_{r+k} < r. \tag{2.6}$$

Since $\overleftarrow{\Delta}_h^r f(x) = (-1)^r\, \overrightarrow{\Delta}_{-h}^r f(x)$, we also have

$$\overleftarrow{\Delta}_h^r P_n(x) = \sum_{k=0}^{n-r} (-1)^k \frac{P_n^{(r+k)}(x)}{k!} h^{r+k} \xi_{r+k}^{k}, \qquad 0 < \xi_{r+k} < r. \tag{2.7}$$

In both (2.6) and (2.7), ξ_{r+k} depends only on r and k.

We will extend our results to the weighted DT modulus of smoothness.

Definition 1 (*Ditzian and Totik [5, Chapter 8]*). A positive weight function w on $(-1,\ 1)$ is of class J_p^* if

(a) $w(x) = w_-(\sqrt{1+x})\ w_+(\sqrt{1-x})$;
(b) $w_+(y) = y^{\gamma_1} v_+(y)$, $w_-(y) = y^{\gamma_2} v_-(y)$, where $\gamma_i > -2/p$ and $v_\pm \sim 1$ on every interval $[\delta,\ \sqrt{2}]$, $\delta > 0$;
(c) for every $\varepsilon > 0$, $y^\varepsilon v_\pm(y)$ are increasing and $y^{-\varepsilon} v_\pm(y)$ are decreasing in $(0,\ \delta(\varepsilon))$ for some $\delta(\varepsilon) > 0$; and
(d) for $p = \infty$ we may have $\gamma_1 = 0$ or $\gamma_2 = 0$ in which case $v_-(y)$ or $v_+(y)$ have to be nondecreasing for small y.

One can see from the definition that J_p^* contains the Jacobi weights $w(x) = (1+x)^{\gamma_1}(1-x)^{\gamma_2}$, $\gamma_i > -1/p$ for $0 < p < \infty$, and $\gamma_i \geqslant 0$ for $p = \infty$; in particular, it contains the constant function $w(x) \equiv 1$. Also, if w is in J_p^*, or is a Jacobi weight, then so is $w\varphi^j$ for any $j \geqslant 0$. The weighted $\mathbf{L}_p$ norm (or quasi-norm) with weight function $w \in J_p^*$ is defined by

$$\|f\|_{w,\mathbf{L}_p(I)} := \|wf\|_{\mathbf{L}_p(I)}.$$

We will shorten $\|f\|_{w,\mathbf{L}_p[-1,\,1]}$ to $\|f\|_{w,p}$. The weighted DT modulus of smoothness is defined by

$$\begin{aligned}\omega_\varphi^r(f,t)_{w,p} := &\sup_{0\leqslant h\leqslant t} \|w\Delta_{h\varphi}^r f\|_{\mathbf{L}_p[-1+2r^2h^2,\,1-2r^2h^2]}\\ &+ \sup_{0\leqslant h\leqslant 2r^2t^2} \|w\overrightarrow{\Delta}_h^r f\|_{\mathbf{L}_p[-1,\,-1+2r^2t^2]}\\ &+ \sup_{0\leqslant h\leqslant 2r^2t^2} \|w\overleftarrow{\Delta}_h^r f\|_{\mathbf{L}_p[1-2r^2t^2,\,1]},\end{aligned} \tag{2.8}$$

where w is a Jacobi weight with $\gamma_i \geqslant 0$. The weighted DT modulus can be defined for a larger class of weights w, but one has to be careful. For some weight functions w, the differences $w\Delta_{h\varphi}^r f$, $w\overleftarrow{\Delta}_h^r f$ or $w\overrightarrow{\Delta}_h^r$ may not be in $\mathbf{L}_p$ even if wf is, (see the first half of §6.1 of [5]). The weighted DT modulus can be defined for all weights $w \in \mathbf{L}_p$ for polynomials though, since polynomials are bounded on any finite interval. For any $w \in J_p^*$ the weighted main-part modulus of smoothness is defined by

$$\Omega_\varphi^r(f,t)_{w,p} := \sup_{0\leqslant h\leqslant t} \|w\Delta_{h\varphi}^r f\|_{\mathbf{L}_p[-1+2r^2h^2,\,1-2r^2h^2]}. \tag{2.9}$$

We will prove our results on weighted DT moduli only for weights in J_p^* if $1 \leqslant p \leqslant \infty$, and for Jacobi weights with $\gamma_i > -1/p$ if $0 < p < 1$. The reason for this is we only have Bernstein and Remez inequalities for these weights, see the conditions on (2.21) and (2.22) later in this section.

It is well known that $\|\cdot\|_{\mathbf{L}_p[a,\,b]}$ is not a norm but a quasi-norm for $0 < p < 1$, that is, in place of the triangular inequality, we only have

$$\|f+g\|_{\mathbf{L}_p[a,\,b]}^p \leqslant \|f\|_{\mathbf{L}_p[a,\,b]}^p + \|g\|_{\mathbf{L}_p[a,\,b]}^p.$$

Properties and inequalities depending on the triangular inequality need to be re-proved for $0 < p < 1$, which often is more difficult, and some of them simply do not hold anymore. For example, (1.4) is not true for $p < 1$ in general [14, Chapter 7]. We collect below some properties of moduli that are also true for $0 < p < 1$ and/or for the DT moduli, the reader is referred to [2, §12.5; 3,5], for references.

$$\omega^r(f,t)_p \leqslant \omega^r(f,\lambda t)_p \leqslant C\omega^r(f,t)_p, \tag{2.10}$$

$$\omega^r_\varphi(f,t)_p \leqslant \omega^r_\varphi(f,\lambda t)_p \leqslant C\omega^r_\varphi(f,t)_p, \tag{2.11}$$

$$\omega^r_\varphi(f,t)_p \leqslant C\|f\|_p, \tag{2.12}$$

$$\omega^r(f+g,t)_p^q \leqslant \omega^r(f,t)_p^q + \omega^r(g,t)_p^q, \tag{2.13}$$

where $0 < p \leqslant \infty$, $\lambda > 1$, $q = \min(1, p)$, and C is a constant depending only on r, q, and also on λ if applicable. The triangular inequality (2.13) also holds if ω^r is replaced by ω^r_φ or Ω^r_φ and/or a weight w is added. For $1 \leqslant p \leqslant \infty$ and $\lambda > 1$ we have

$$\Omega^r_\varphi(f,t)_{w,p} \leqslant \Omega^r_\varphi(f,\lambda t)_{w,p} \leqslant C\lceil\lambda\rceil^r \Omega^r_\varphi(f,t)_{w,p}, \quad w \in J_p^*, \tag{2.14}$$

$$\Omega^r_\varphi(f,t)_{w,p} \leqslant C\|wf\|_{\mathbf{L}_p[-1+2r^2t^2,\,1-2r^2t^2]}, \quad w \in J_p^*, \tag{2.15}$$

$$\begin{gathered}\omega^r_\varphi(f,t)_{w,p} \leqslant \omega^r_\varphi(f,\lambda t)_{w,p} \leqslant C\lceil\lambda\rceil^r \omega^r_\varphi(f,t)_{w,p},\\ w \text{ is a Jacobi weight with } \gamma_i \geqslant 0,\end{gathered} \tag{2.16}$$

where C depends on r and the weight w. These inequalities can be deduced from their equivalence to the respective K-functionals [5, Chapters 8 and 6], namely

$$\begin{aligned}\Omega^r_\varphi(f,t)_{w,p} &\sim \mathcal{K}_{r,\varphi}(f,t^r)_{w,p}\\ &:= \sup_{0<h\leqslant t}\inf_g\{\|w(f-g)\|_{\mathbf{L}_p[-1+2r^2h^2,\,1-2r^2h^2]}\\ &\qquad + h^r\|w\varphi^r g^{(r)}\|_{\mathbf{L}_p[-1+2r^2h^2,\,1-2r^2h^2]} : g^{(r-1)} \in \mathrm{AC}[-1+2r^2h^2,\ 1-2r^2h^2]\}\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\omega^r_\varphi(f,t)_{w,p} &\sim K_{r,\varphi}(f,t^r)_{w,p}\\ &:= \inf\{\|w(f-g)\|_p + t^r\|w\varphi^r g^{(r)}\|_p : g^{(r-1)} \in \mathrm{AC_{loc}}[-1,\ 1]\}.\end{aligned} \tag{2.18}$$

Several types of inequalities are of fundamental importance in polynomial approximation, namely Bernstein-, Markov- and Remez-type inequalities. The Bernstein inequality for algebraic polynomials takes the form

$$\|\varphi P_n'\|_p \leqslant n\|P_n\|_p, \quad P_n \in \mathcal{P}_n, \ \ 0 < p \leqslant \infty.$$

Markov's inequality (see [1, Theorem A.4.14] for a more general version) has the form

$$\|P_n'\|_p \leqslant Cn^2\|P_n\|_p, \quad 0 < p \leqslant \infty, \tag{2.19}$$

where C can be written as $A^{1+1/p}$ with A an absolute constant. The Remez inequality, (see [15] for $p = \infty$ and [1] for $0 < p < \infty$), is given in the following lemma:

Lemma B. *For any $P_n \in \mathcal{P}_n$, any measurable $A \subseteq [-1, 1]$ with a Lebesgue measure $2 - an^{-2}$ for some $0 \leqslant a \leqslant n^2/2$, and $0 < p \leqslant \infty$ we have*

$$\|P_n\|_p \leqslant C \|P_n\|_{\mathbf{L}_p(A)}, \tag{2.20}$$

where C depends on a and $q = \min(1, p)$.

We will also need weighted Bernstein- and Remez-type inequalities. The Bernstein inequality we will need is

$$\|w\varphi P_n'\|_p \leqslant Cn\|wP_n\|_p, \quad P_n \in \mathcal{P}_n, \tag{2.21}$$

where C depends on w and q (again $q = \min(1, p)$ throughout the paper), $w \in J_p^*$ if $p \geqslant 1$ ([5, Theorem 8.4.7]), and w is any Jacobi weight with $\gamma_i > -1/p$ if $0 < p < 1$, (a special case of Nevai [13, Theorem 5] in which one chooses the number of nodes $N = 2$ and the exponents $\Gamma_1 = \Gamma_2 = 0$). We remind the reader that Jacobi weights belong to J_p^* if $\gamma_i > -1/p$ for $0 < p < \infty$, and $\gamma_i \geqslant 0$ for $p = \infty$. The weighted Remez inequality we will need is

$$\|wP_n\|_p \leqslant C\|wP_n\|_{\mathbf{L}_p[-1+an^{-2},\, 1-an^{-2}]}, \qquad n^2 > a \geqslant 0, \quad P_n \in \mathcal{P}_n. \tag{2.22}$$

where C depends on w, a and q, $w \in J_p^*$ if $p \geqslant 1$ [5, Theorem 8.4.8], and w is any Jacobi weight with $\gamma_i > -1/p$ if $0 < p < 1$, which was proved by Nevai [12, Chapter 6, Theorem 14] for $0 < p < \infty$ in a different form.

One key step in dealing with ω_φ^r is to estimate $\|w\varphi^k P_n^{(k)}\|$ with k being as large as n. We could use (2.21) with $w\varphi^{k-1}$ as the weight, but this way the constant C would depend on k thus also on n, which is unacceptable. For this reason, we need the following two special versions of (2.21), whose constant is independent of k and n. One of them is for the non-weighted ($w \equiv 1$) DT modulus [4, 2.3]:

$$\|\varphi^k P_n'\|_p \leqslant Cnk\, \|\varphi^{k-1} P_n\|_p, \quad 0 < p \leqslant \infty, \quad 1 \leqslant k \leqslant n, \tag{2.23}$$

where C depends only on q. The other one is for the weighted DT modulus:

$$\|w\varphi^k P_n'\|_p \leqslant Cnk\, \|w\varphi^{k-1} P_n\|_p, \quad 0 < p \leqslant \infty, \quad 1 \leqslant k \leqslant n, \tag{2.24}$$

where $w \in J_p^*$ for $p \geqslant 1$, and is any Jacobi weight with $\gamma_i > -1/p$ for $0 < p < 1$, and C depends on w and q. This can be proved in a way almost identical to that of (2.23), see [4]. The proof will use (2.21) with w replaced by $\omega\varphi^{k-2\lfloor k/2\rfloor-1}$, and (2.22) with w replaced by $w\varphi^{k-2\lfloor k/2\rfloor}$. Note that $k - 2\lfloor k/2\rfloor$ equals either 0 or 1. This is why the constant C is independent of k.

Remark. It is because of the presence of φ in Bernstein inequalities that our results on the DT and weighted DT moduli are only proved for $\varphi(x) = \sqrt{1-x^2}$.

3. Main results

Using a technique adopted from [3], we first generalize the result of Zhou and Zhou [19] to the $\mathbf{L}_p$ space. Recall from (1.9) that $q = \min(p, 1)$ throughout this paper.

Theorem 2. *Let $P_n \in \mathcal{P}_n[-1, 1]$, $n \geqslant 1$, $r \geqslant 1$ and $0 < p \leqslant \infty$. Then*

$$\omega^r(P_n, t)_p \sim t\, \omega^{r-1}(P_n', t)_p \sim \cdots \sim t^r \| P_n^{(r)} \|_p, \quad 0 \leqslant t \leqslant n^{-2}, \tag{3.1}$$

where the equivalence constants depend only on r and q.

Proof. In view of (2.10) we can assume $0 \leqslant h \leqslant t \leqslant t_0 := 1/(AC_0 r n^2)$, where C_0 is the constant in Markov's inequality (2.19), and $A \geqslant 1$ is chosen so that $\sum_{k=1}^{\infty} \frac{1}{(A^k k!)^q} \leqslant \frac{1}{2}$. It suffices to show $\omega^r(P_n, t)_p \sim t^r \| P_n^{(r)} \|_p$ only, since $t^j\, \omega^{r-j}(P_n^{(j)}, t)_p \sim t^r \| P_n^{(r)} \|_p$ follows from this by replacing r by $r - j$ and replacing P_n by $P_n^{(j)}$. Using (2.6) and (2.19) we obtain

$$\begin{aligned} \| \overrightarrow{\Delta}_h^r P_n \|_{\mathbf{L}_p[-1,\,1-rh]}^q &\leqslant \| \overrightarrow{\Delta}_h^r P_n \|_p^q \leqslant h^{qr} \sum_{k=0}^{n-r} \left(\frac{\| P_n^{(r+k)} \|_p}{A^k C_0^k n^{2k} k!} \right)^q \\ &\leqslant t^{qr} \| P_n^{(r)} \|_p^q \left(1 + \sum_{k=1}^{n-r} \frac{1}{(A^k k!)^q} \right) \leqslant \frac{3t^{qr}}{2} \| P_n^{(r)} \|_p^q. \end{aligned}$$

This shows $\omega^r(P_n, t)_p \leqslant (\frac{3}{2})^{1/q} t^r \| P_n^{(r)} \|_p$. Similarly, by (2.20) and (2.19)

$$\begin{aligned} C^q \| \overrightarrow{\Delta}_t^r P_n \|_{\mathbf{L}_p[-1,\,1-rt]}^q &\geqslant \| \overrightarrow{\Delta}_t^r P_n \|_p^q \geqslant t^{qr} \| P_n^{(r)} \|_p^q - t^{qr} \sum_{k=1}^{n-r} \left(\frac{\| P_n^{(r+k)} \|_p}{A^k C_0^k n^{2k} k!} \right)^q \\ &\geqslant t^{qr} \| P_n^{(r)} \|_p^q \left(1 - \sum_{k=1}^{\infty} \frac{1}{(A^k k!)^q} \right) \geqslant \frac{t^{qr}}{2} \| P_n^{(r)} \|_p^q, \end{aligned}$$

thus $C\omega^r(P_n, t)_p \geqslant C \| \overrightarrow{\Delta}_t^r P_n \|_{\mathbf{L}_p[-1,\,1-rt]} \geqslant t^r \| P_n^{(r)} \|_p$. $\square$

Ditzian et al. showed in the proof of [3, Lemma 5.4] these two inequalities:

$$\| \Delta_{h\varphi}^r P_n \|_{\mathbf{L}_p(I_{rh})} \leqslant C h^r \| \varphi^r P_n^{(r)} \|_p, \tag{3.2a}$$

$$h^r \| \varphi^r P_n^{(r)} \|_p \leqslant C \| \Delta_{h\varphi}^r P_n \|_{\mathbf{L}_p(I_{rh})}, \tag{3.2b}$$

which imply part of the following theorem, namely $\omega_\varphi^r(f, t)_p \sim t^r \| P_n^{(r)} \|_{\varphi^r, p}$. We will still give a proof, somewhat simpler and more straightforward, to this part for completeness, and also because we will need to modify it for other parts of the theorem. We point out that inequalities in both directions are needed to establish the equivalence, since even if $p \geqslant 1$, the equivalent of (1.4)

$$\omega_\varphi^r(f, t)_p \leqslant C t\, \omega_\varphi^{r-1}(f', t)_{\varphi, p}$$

is not known in general for the step-weight $\varphi(x) = \sqrt{1 - x^2}$ we use in this paper, although it is known for some other step-weight functions φ (see [5, Corollary 6.3.3]).

Theorem 3. *Let $r \geqslant 1$, $n \geqslant 1$, $0 \leqslant t \leqslant n^{-1}$ and $0 < p \leqslant \infty$. Then for any $P_n \in \mathcal{P}_n$*

$$\omega_\varphi^r(P_n, t)_p \sim t\omega_\varphi^{r-1}(P_n', t)_{\varphi,p} \sim t^2\omega_\varphi^{r-2}(P_n'', t)_{\varphi^2,p} \sim \cdots \sim t^r \|P_n^{(r)}\|_{\varphi^r,p}, \tag{3.3}$$

with the equivalence constants depending only on r and q.

Proof. In view of (2.11) and (2.16) we can assume $0 \leqslant t \leqslant t_0 := \min\left(\frac{1}{AC_1r^2n}, \frac{1}{2\sqrt{AC_1}\,r^2n}\right)$, where C_1 is the constant in (2.23), and $A = 3^{1/2q}$ (so that $\sum_{k=1}^\infty \frac{1}{A^{2kq}} = 1/2$). We first prove $\omega_\varphi^r(P_n, t)_p \sim t^r \|\varphi^r P_n^{(r)}\|_p$. Using (2.23) and recalling $|\xi_j| < r/2$ in (2.3) and (2.4), we have for any $0 \leqslant h \leqslant t \leqslant t_0$

$$\begin{aligned}\frac{\|\varphi^{r+k}P_n^{(r+k)}\|_p}{k!}h^k|\xi_{r+k}|^k &\leqslant C_1n(r+k)\frac{\|\varphi^{r+k-1}P_n^{(r+k-1)}\|_p}{k!}\,\frac{h^{k-1}}{AC_1r^2n}\,\frac{r|\xi_{r+k}|^{k-1}}{2}\\ &= \frac{\|\varphi^{r+k-1}P_n^{(r+k-1)}\|_p}{(k-1)!A}\,\frac{(r+k)h^{k-1}|\xi_{r+k}|^{k-1}}{2kr}\\ &\leqslant \frac{\|\varphi^{r+k-1}P_n^{(r+k-1)}\|_p}{(k-1)!A}\,h^{k-1}|\xi_{r+k}|^{k-1} \leqslant \cdots\\ &\leqslant \frac{\|\varphi^r P_n^{(r)}\|_p}{A^k},\end{aligned}$$

where we have used the fact $1/k + 1/r \leqslant 2$ or $r + k \leqslant 2rk$ for $r, k \geqslant 1$. Now by (2.4)

$$\begin{aligned}\|\Delta_{h\varphi}^r P_n\|_{\mathbf{L}_p(I_{rh})}^q &\leqslant \|\Delta_{h\varphi}^r P_n\|_p^q \leqslant h^{rq}\sum_{k=0}^K\left[\frac{\|\varphi^{r+2k}P_n^{(r+2k)}\|_p}{(2k)!}h^{2k}|\xi_{r+2k}|^{2k}\right]^q\\ &\leqslant h^{rq}\|\varphi^r P_n^{(r)}\|_p^q\left(1+\sum_{k=1}^K\frac{1}{A^{2kq}}\right) \leqslant \frac{3t^{rq}}{2}\|\varphi^r P_n^{(r)}\|_p^q.\end{aligned} \tag{3.4}$$

Therefore

$$\omega_\varphi^r(P_n, t)_p \leqslant (3/2)^{1/q} t^r \|\varphi^r P_n^{(r)}\|_p. \tag{3.5}$$

On the other hand, because of (2.5) we can apply to $\Delta_{h\varphi}^r P_n$ either (2.20), or (2.22) with $w = \varphi$, and obtain

$$\begin{aligned}C^q\|\Delta_{h\varphi}^r P_n\|_{\mathbf{L}_p(I_{rh})}^q &\geqslant \|\Delta_{h\varphi}^r P_n\|_p^q \geqslant h^{rq}\|\varphi^r P_n^{(r)}\|_p^q\left(1-\sum_{k=1}^K\frac{1}{A^{2kq}}\right)\\ &\geqslant \frac{h^{rq}}{2}\|\varphi^r P_n^{(r)}\|_p^q.\end{aligned}$$

Therefore

$$C\omega_\varphi^r(P_n, t)_p \geqslant C\|\Delta_{t\varphi}^r P_n\|_{\mathbf{L}_p(I_{rt})} \geqslant \frac{t^r}{2^{1/q}}\|\varphi^r P_n^{(r)}\|_p, \tag{3.6}$$

hence $\omega_\varphi^r(P_n, t)_p \sim t^r \|\varphi^r P_n^{(r)}\|_p = t^r \|P_n^{(r)}\|_{\varphi^r,p}$.

We now show $t^j\omega_\varphi^{r-j}(P_n^{(j)},t)_{\varphi^j,p}\sim t^r\|P_n^{(r)}\|_{\varphi^r,p}$ for $1\leqslant j<r$. Replacing in (2.4) r by $r-j$ and P_n by $P_n^{(j)}$ (whose degree is $n-j$), and multiplying both sides by $t^j\varphi(x)^j$ give

$$t^j\varphi(x)^j\Delta_{h\varphi(x)}^{r-j}P_n^{(j)}(x)=t^jh^{r-j}\sum_{k=0}^{K}\frac{\varphi(x)^{r+2k}P_n^{(r+2k)}(x)}{(2k)!}h^{2k}\xi_{r-j+2k}^{2k},$$

where $K=\lfloor\frac{(n-j)-(r-j)}{2}\rfloor=\lfloor\frac{n-r}{2}\rfloor$. By almost the same arguments as those used in (3.6)

$$\begin{aligned}Ct^j\omega_\varphi^{r-j}(P_n^{(j)},t)_{\varphi^j,p}&\geqslant Ct^j\sup_{0\leqslant h\leqslant t}\|\varphi^j\Delta_{h\varphi}^{r-j}P_n^{(j)}\|_{\mathbf{L}_p[-1+2(r-j)^2h^2,\,1-2(r-j)^2h^2]}\\&\geqslant\frac{t^r}{2^{1/q}}\|\varphi^rP_n^{(r)}\|_p.\end{aligned}$$

For the inequality in the other direction, we estimate separately the three terms in (2.8), the definition of the weighted DT modulus. For the first term, it is similar to (3.4) and (3.5):

$$t^j\sup_{0\leqslant h\leqslant t}\|\varphi^j\Delta_{h\varphi}^{r-j}P_n^{(j)}\|_{\mathbf{L}_p[-1+2(r-j)^2h^2,\,1-2(r-j)^2h^2]}\leqslant(3/2)^{1/q}t^r\|\varphi^rP_n^{(r)}\|_p.$$

For the second term in the definition, we replace in (2.6) r by $r-j$ and P_n by $P_n^{(j)}$, multiply both sides by $t^j\varphi(x)^j$ and obtain

$$t^j\varphi(x)^j\overrightarrow{\Delta}_h^{r-j}P_n^{(j)}(x)=t^jh^{r-j}\varphi(x)^j\sum_{k=0}^{n-r}\frac{P_n^{(r+k)}(x)}{k!}h^k\xi_{r-j+k}^k,$$

where $0<\xi_{r-j+k}<r-j$. Because the supremum in this term is taken over all h such that $0\leqslant h\leqslant 2(r-j)^2t^2$, thus $0<h\leqslant 1/(2AC_1r^2n^2)$. By the fact that $1/n<\varphi(x)$ for $x\in[-1+n^{-2},\,1-n^{-2}]$, and by (2.22) and (2.23)

$$\begin{aligned}&t^{jq}\|\varphi^j\overrightarrow{\Delta}_h^{r-j}P_n^{(j)}\|^q_{\mathbf{L}_p[-1,\,-1+2(r-j)^2t^2]}\\&\leqslant C^qt^{(2r-j)q}\left\|\sum_{k=0}^{n-r}\frac{\varphi^jP_n^{(r+k)}}{k!}h^k\xi_{r-j+k}^k\right\|^q_{\mathbf{L}_p[-1,\,-1+2(r-j)^2t^2]}\\&\leqslant\frac{C^qt^{rq}}{n^{(r-j)q}}\left\|\sum_{k=0}^{n-r}\frac{\varphi^jP_n^{(r+k)}}{k!}h^k\xi_{r-j+k}^k\right\|^q_{\mathbf{L}_p[-1+n^{-2},\,1-n^{-2}]}\\&\leqslant C^qt^{rq}\sum_{k=0}^{n-r}\left[\frac{\|\varphi^rP_n^{(r+k)}\|_{\mathbf{L}_p[-1+n^{-2},\,1-n^{-2}]}}{(2AC_1rn^2)^kk!}\right]^q\\&\leqslant C^qt^{rq}\sum_{k=0}^{n-r}\left[\frac{\|\varphi^{r+k}P_n^{(r+k)}\|_{\mathbf{L}_p[-1+n^{-2},\,1-n^{-2}]}}{(2AC_1rn)^kk!}\right]^q\\&\leqslant C^qt^{rq}\|\varphi^rP_n^{(r)}\|_p^q\sum_{k=0}^{\infty}\frac{1}{A^{kq}}\leqslant C^qt^{rq}\|\varphi^rP_n^{(r)}\|_p^q.\end{aligned}$$

Taking the supremum of the left side of this gives

$$t^j \sup_{0 \leqslant h \leqslant 2(r-j)^2 t^2} \|\varphi^j \overrightarrow{\Delta}_h^{r-j} P_n^{(j)}\|_{\mathbf{L}_p[-1,\,-1+2(r-j)^2t^2]} \leqslant C t^r \|\varphi^r P_n^{(r)}\|_p.$$

The proof of

$$t^j \sup_{0 \leqslant h \leqslant 2(r-j)^2 t^2} \|\varphi^j \overleftarrow{\Delta}_h^{r-j} P_n^{(j)}\|_{\mathbf{L}_p[1-2(r-j)^2t^2,\,1]} \leqslant C t^r \|\varphi^r P_n^{(r)}\|_p$$

is almost identical. Now we have shown $t^j \omega_\varphi^{r-j}(P_n^{(j)}, t)_{\varphi^j,p} \sim t^r \|\varphi^r P_n^{(r)}\|_p$, $0 \leqslant j < r$, and finished the proof of the theorem. $\square$

We observe that arguments almost identical to those in the second part of the above proof (with (2.23) replaced by (2.24)) will show

$$t^j \omega_\varphi^{r-j}(P_n^{(j)}, t)_{w\varphi^j,p} \sim t^r \|P_n^{(r)}\|_{w\varphi^r,p}, \quad 0 \leqslant j < r,$$

that is,

Theorem 4. *Let* $r \geqslant 1$, $n \geqslant 1$, $0 \leqslant t \leqslant (Mn)^{-1}$ *and* $0 < p \leqslant \infty$, *and let* w *be in* J_p^* *if* $p \geqslant 1$ *and be a Jacobi weight with* $\gamma_i \geqslant -1/p$ *if* $0 < p < 1$, *then for any* $P_n \in \mathcal{P}_n$

$$\begin{aligned} \omega_\varphi^r(P_n, t)_{w,p} &\sim t\omega_\varphi^{r-1}(P_n', t)_{w\varphi,p} \sim t^2\omega_\varphi^{r-2}(P_n'', t)_{w\varphi^2,p} \\ &\sim \cdots \sim t^r \|P_n^{(r)}\|_{w\varphi^r,p}, \end{aligned} \tag{3.7}$$

where M and the equivalence constants depending on r, q and the weight w. *If* $p \geqslant 1$ *and* w *is a Jacobi weight with* $\gamma_i \geqslant 0$, *then one can take* $M = 1$.

Remark. The reason for the constant M in the theorem is that (2.16) is only known for $p \geqslant 1$ and Jacobi weights with $\gamma_i \geqslant 0$. Similarly, the reason for the constant M in Theorem 5 below is the restriction $p \geqslant 1$ on inequality (2.14).

The theorem is also valid if the weighted modulus $\omega_\varphi^r(P_n, t)_{w,p}$ is replaced by the main-part modulus $\Omega_\varphi^r(P_n, t)_{w,p}$ defined by (2.9), as stated below. We leave the proof to the reader.

Theorem 5. *Let* $r \geqslant 1$, $n \geqslant 1$, $0 \leqslant t \leqslant (Mn)^{-1}$ *and* $0 < p \leqslant \infty$, *and let* w *be in* J_p^* *if* $p \geqslant 1$ *and be a Jacobi weight with* $\gamma_i > -1/p$ *if* $0 < p < 1$, *then for any* $P_n \in \mathcal{P}_n$

$$\begin{aligned} \Omega_\varphi^r(P_n, t)_{w,p} &\sim t\Omega_\varphi^{r-1}(P_n', t)_{w\varphi,p} \sim t^2\Omega_\varphi^{r-2}(P_n'', t)_{w\varphi^2,p} \\ &\sim \cdots \sim t^r \|P_n^{(r)}\|_{w\varphi^r,p}, \end{aligned} \tag{3.8}$$

where M and the equivalence constants depending on r, q and the weight w. If $p \geqslant 1$ *then one can take* $M = 1$.

The following corollary says the main-part moduli are also equivalent to the "whole" moduli ω_φ^r for polynomials.

Corollary 6. *Under the conditions of Theorem* 5 *we have*

$$\Omega_\varphi^r(P_n, t)_{w,p} \sim \omega_\varphi^r(P_n, t)_{w,p}, \quad 0 \leqslant t \leqslant (Mn)^{-1}. \tag{3.9}$$

In particular, if $w(x) \equiv 1$, *we have*

$$\Omega_\varphi^r(P_n, t)_p \sim \omega_\varphi^r(P_n, t)_p, \quad 0 \leqslant t \leqslant (Mn)^{-1}. \tag{3.10}$$

4. Asymptotic behavior of best approximating polynomials

In this section, we give two examples to show the usefulness of the equivalence in applications. In the first example, P_n^* denotes a best approximation to f in $\mathbf{L}_p$ from $\mathcal{P}_n$, and $E_n(f)_p := \|f - P_n^*\|_p$. Section 7.3 of [5] is devoted to asymptotic behavior of derivatives of best approximating polynomials. The final result of the section is

Theorem C. *For* $0 < \alpha \leqslant r$ *and* $1 \leqslant p \leqslant \infty$, $\|\varphi^r P_n^{*(r)}\|_p = \mathcal{O}(n^{r-\alpha})$ *and* $\omega_\varphi^r(f, n^{-1})_p = \mathcal{O}(n^{-\alpha})$ *are equivalent.*

As an application, we prove the following generalization of Theorem C, which is more balanced and easier to prove, and holds for $0 < p < 1$ as well.

Theorem 7. *For* $0 < \alpha \leqslant r$ *and* $0 < p \leqslant \infty$, $\omega_\varphi^r(P_n^*, n^{-1})_p = \mathcal{O}(n^{-\alpha})$ *and* $\omega_\varphi^r(f, n^{-1})_p = \mathcal{O}(n^{-\alpha})$ *are equivalent, where the equivalence constants depend on r and* $q = \min(1, p)$, *and also on* α *if* α *is close to zero.*

This theorem is a direct consequence of the next lemma, which is a modification (and an extension to $0 < p \leqslant \infty$) of Theorems 7.3.1 and 7.3.2 of [5]. We changed the formulation of Theorem 7.3.2, but merely replaced $n^{-r}\|\varphi^r P_n^{*(r)}\|_p$ by $\omega_\varphi^r(P_n^*, n^{-1})_p$ in Theorem 7.3.1, which says $\|\varphi^r P_n^{*(r)}\|_p \leqslant Cn^r \omega_\varphi^r(f, n^{-1})_p$. Without using $\omega_\varphi^r(P_n^*, n^{-1})_p \sim n^{-r}\|\varphi^r P_n^{*(r)}\|_p$, its proof is much more than trivial.

Lemma 8. *For* $0 < p \leqslant \infty$

$$n^{-r}\|\varphi^r P_n^{*(r)}\|_p \sim \omega_\varphi^r(P_n^*, n^{-1})_p \leqslant C\omega_\varphi^r(f, n^{-1})_p, \tag{4.1}$$

$$\omega_\varphi^r(f, t)_p \leqslant C\left[\sum_{k=1}^{\infty} \omega_\varphi^r(P_{2^k n}^*, 2^{-k}n^{-1})_p^q\right]^{1/q}, \quad 0 < t \leqslant 1, \quad n = [t^{-1}], \tag{4.2}$$

where C depends only on r and q.

Proof. (4.1) follows from a standard argument:

$$\omega_\varphi^r(f - P_n^*, n^{-1})_p \leqslant C\|f - P_n^*\|_p = CE_n(f)_p \leqslant C\omega_\varphi^r(f, n^{-1})_p$$

and

$$\omega_\varphi^r(P_n^*, n^{-1})_p^q \leqslant \omega_\varphi^r(f - P_n^*, n^{-1})_p^q + \omega_\varphi^r(f, n^{-1})_p^q.$$

For (4.2) we use the idea of Sunouchi [17] as Ditzian and Totik did in [5]. For any $n \geqslant 1$, let $\bar{P}_n(P^*_{2n})$ be a best approximation to P^*_{2n} in $\mathbf{L}_p$ from $\mathcal{P}_n$, then

$$I_n := \|P^*_{2n} - \bar{P}_n(P^*_{2n})\|_p = E_n(P^*_{2n})_p \leqslant C\omega^r_\varphi(P^*_{2n}, n^{-1})_p \leqslant C\omega^r_\varphi(P^*_{2n}, (2n)^{-1})_p$$

and

$$I^q_n \geqslant \|f - \bar{P}_n(P^*_{2n})\|^q_p - \|f - P^*_{2n}\|^q_p \geqslant E_n(f)^q_p - E_{2n}(f)^q_p.$$

We can now write

$$E_n(f)^q_p = \sum_{k=0}^{\infty}\left(E_{2^k n}(f)^q_p - E_{2^{k+1}n}(f)^q_p\right) \leqslant \sum_{k=0}^{\infty} I^q_{2^k n} \leqslant C^q \sum_{k=1}^{\infty} \omega^r_\varphi(P^*_{2^k n}, 2^{-k}n^{-1})^q_p.$$

For any $0 < t \leqslant 1$ let $n = [t^{-1}]$. Then

$$\begin{aligned}
\omega^r_\varphi(f,t)^q_p &\leqslant C^q\omega^r_\varphi(f,(2n)^{-1})^q_p \\
&\leqslant C^q[\omega^r_\varphi(f - P^*_{2n}, (2n)^{-1})^q_p + \omega^r_\varphi(P^*_{2n}, (2n)^{-1})^q_p] \\
&\leqslant C^q[E_{2n}(f)^q_p + \omega^r_\varphi(P^*_{2n}, (2n)^{-1})^q_p] \\
&\leqslant C^q \sum_{k=1}^{\infty} \omega^r_\varphi(P^*_{2^k n}, 2^{-k}n^{-1})^q_p. \qquad \square
\end{aligned}$$

In our second example we let $1 \leqslant p \leqslant \infty$, $w \in J^*_p$, P^*_n be a best weighted approximation to f in $\mathbf{L}_p$ from $\mathcal{P}_n$ and $E_n(f)_{w,p} := \|f - P^*_n\|_{w,p}$. We also let $D_{rn} := [-1 + 2r^2/n^2,\ 1 - 2r^2/n^2]$ and $\tilde{E}_n(f)_{w,p} := \|f - P^*_n\|_{w,\mathbf{L}_p(D_{rn})}$. This example is about an analog to Theorem C (see §8.3 of [5]):

Theorem D. *If $1 \leqslant p \leqslant \infty$, $w \in J^*_p$ and $0 < \alpha \leqslant r$, then*

$$\|P^{*(r)}_n\|_{w\varphi^r,p} \leqslant Cn^r \int_0^{1/n} \Omega^r_\varphi(f,\tau)_{w,p}\tau^{-1}d\tau, \tag{4.3}$$

$$\Omega^r_\varphi(f,t)_{w,p} \leqslant C \sum_{k=1}^{\infty} 2^{-kr}n^{-r}\|P^{*(r)}_{2^k n}\|_{w\varphi^r,p}, \quad n = [t^{-1}]. \tag{4.4}$$

As a consequence, the conditions $\|P^{*(r)}_n\|_{w\varphi^r,p} = \mathcal{O}(n^{r-\alpha})$ and $\Omega^r_\varphi(f,t)_{w,p} = \mathcal{O}(t^\alpha)$ are equivalent.

The complex form of (4.3) comes from (8.2.1) of [5]

$$E_n(f)_{w,p} \leqslant C \sum_{k=0}^{\infty} \Omega^r_\varphi(f, 2^{-k}n^{-1})_{w,p} \sim \int_0^{1/n} \Omega^r_\varphi(f,\tau)_{w,p}\tau^{-1}\,d\tau,$$

whose complexity is understandable since it bounds the approximation error $E_n(f)_{w,p}$ on the whole interval $[-1,\ 1]$ by the *main-part* modulus of f. This is another situation in which our newly proved equivalence $\Omega_\varphi^r(P_n^*, n^{-1})_{w,p} \sim n^{-r}\|P_n^{*(r)}\|_{w\varphi^r,p}$ can help, bridging $[-1,\ 1]$ and its subinterval D_{rn} and resulting in an inequality stronger than (4.3). We have

Theorem E. *If* $1 \leqslant p \leqslant \infty$, $w \in J_p^*$ *and* $0 < \alpha \leqslant r$, *then*

$$\Omega_\varphi^r(P_n^*, n^{-1})_{w,p} \leqslant C\Omega_\varphi^r(f, n^{-1})_{w,p}, \tag{4.5}$$

$$\Omega_\varphi^r(f, t)_{w,p} \leqslant C\sum_{k=1}^{\infty}\Omega_\varphi^r(P_{2^k n}^*, 2^{-k}n^{-1})_{w\varphi^r,p}, \quad 0 < t \leqslant 1, \quad n = [t^{-1}]. \tag{4.6}$$

As a consequence, the conditions $\Omega_\varphi^r(P_n^*, n^{-1})_{w,p} = \mathcal{O}(n^{-\alpha})$ *and* $\Omega_\varphi^r(f, t)_{w,p} = \mathcal{O}(t^\alpha)$ *are equivalent.*

The proofs of (4.5) and (4.6) are very similar to those of (4.1) and (4.2), in which one needs the Jackson inequality on D_{rn} ((8.2.4) in [5])

$$\tilde{E}_n(f)_p = \|f - P_n^*\|_{w,\mathbf{L}_p(D_{rn})} \leqslant C\Omega_\varphi^r(f, n^{-1})_{w,p}, \tag{4.7}$$

inequalities (2.14) and (2.15), and a variation of (2.13) for $\Omega_\varphi^r(\cdot,\cdot)_{w,p}$.

We conclude the paper by a comment on part (a) of Remark 7.3.4 of [5]. If $n^{-r}\|\varphi^r P_n^{*(r)}\|_p$ is replaced by $\omega_\varphi^r(P_n^*, n^{-1})_p$, these interesting statements on the relationship among the orders of $E_n(f)_p$, $\omega_\varphi^r(f, n^{-1})_p$ and $n^{-r}\|\varphi^r P_n^{*(r)}\|_p$ (to be replaced by $\omega_\varphi^r(P_n^*, n^{-1})_p$) will be more natural and balanced, thus will be even more interesting.

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References

[1] P. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York, 1995.
[2] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.
[3] Z. Ditzian, V.H. Hristov, K.G. Ivanov, Moduli of smoothness and K-functionals in $\mathbf{L}_p$, $0 < p < 1$, Constr. Approx. 11 (1995) 67–83.
[4] Z. Ditzian, D. Jiang, D. Leviatan, Inverse theorem for best polynomial approximation in $\mathbf{L}_p$, $0 < p < 1$, Proc. Amer. Math. Soc. 120 (1994) 151–155.
[5] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, New York, 1987.
[6] Y.-K. Hu, On equivalence of moduli of smoothness, J. Approx. Theory 97 (1999) 282–293.
[7] Y.-K. Hu, K.A. Kopotun, X.M. Yu, On positive and copositive polynomial and spline approximation in $\mathbf{L}_p[-1,\ 1]$, $0 < p < \infty$, J. Approx. Theory 86 (1996) 320–334.

[8] Y.-K. Hu, D. Leviatan, X.M. Yu, Copositive polynomial and spline approximation, J. Approx. Theory 80 (1995) 204–218.
[9] Y.-K. Hu, X.M. Yu, Discrete modulus of smoothness of splines with equally spaced knots, SIAM J. Num. Anal. 32 (1995) 1428–1435.
[10] Y.-K. Hu, X.M. Yu, The degree of copositive approximation and a computer algorithm, SIAM J. Num. Anal. 33 (1996) 388–398.
[11] Y.-K. Hu, X.M. Yu, Copositive polynomial approximation revisited, in: G.A. Anastassiou (Ed.), Applied Mathematics Reviews, vol. 1, World Scientific, Singapore, 2000, pp. 157–174.
[12] P.G. Nevai, Orthogonal polynomials, Mem. Amer. Math. Soc. 18 (213) (1979).
[13] P.G. Nevai, Bernstein's inequality in $\mathbf{L}_p$ for $0 < p < 1$, J. Approx. Theory 27 (1979) 239–243.
[14] P.P. Petrushev, V.A. Popov, Rational Approximation of Real Functions, Cambridge University Press, New York, 1987.
[15] E.J. Remez, Sur une propriété des polynômes de Tchebycheff, Comm. Inst. Sci. Kharkov 13 (1936) 93–95.
[16] S.B. Stechkin, A generalization of some inequalities of S. N. Bernstein, Dokl. Akad. Nauk SSSR 60 (1948) 1511–1514 (in Russian).
[17] G.I. Sunouchi, Derivatives of polynomials of best approximation, in: P.L. Butzer, B. Sz-Nagy (Eds.), Abstract Spaces and Approximation, Proceedings of the Conference on Oberwolfach, 1968, ISNM, vol. 10, Birkhauser, 1969, pp. 233–241.
[18] X.M. Yu, S.P. Zhou, On monotone spline approximation, SIAM J. Math. Anal. 25 (1994) 1227–1239.
[19] G.Z. Zhou, S.P. Zhou, Some remarks on equivalence of moduli of smoothness, J. Approx. Theory 113 (2001) 165–171.